3 The Damped Harmonic Oscillator

3.1 DAMPED HARMONIC MOTION

Various frictional forces may act on a harmonic oscillator, tending to reduce its successive amplitudes. Such a motion is called damped harmonic motion. Suppose a particle of mass m is subject to a restoring force proportional to the distance from a fixed point on the x-axis and a damping force proportional to the velocity. Then the equation of motion becomes

$$m\ddot{x} = -kx - \beta\dot{x} \quad \text{...(3.1)}$$

x being the displacement of the particle from the fixed point at any instant t, k and β are positive constants. The damping force is $-\beta\dot{x}$ where β is the damping coefficient. Equation (3.1) can be written as

$$\ddot{x} + 2b\dot{x} + \omega^2 x = 0 \quad \text{...(3.2)}$$

where $2b = \beta/m$ and $\omega = \sqrt{k/m}$ is the natural frequency of the oscillator. The relaxation time τ is defined by

$$\tau = \frac{1}{2b} = \frac{m}{\beta}. \quad \text{...(3.3)}$$

3.2 DAMPED LC OSCILLATIONS (LCR CIRCUIT)

If resistance R is present in an LC circuit, the total energy

$$U = \frac{1}{2}Li^2 + \frac{q^2}{2C} \quad \text{...(3.4)}$$

is no longer constant, but decreases with time as it is transformed steadily to thermal energy in the resistor:

$$\frac{dU}{dt} = -i^2R$$

Hence,
$$Li\frac{di}{dt} + \frac{q}{C}\frac{dq}{dt} = -i^2R$$

Substituting $i = \dfrac{dq}{dt}$ and $\dfrac{di}{dt} = \dfrac{d^2q}{dt^2}$ and dividing by i, we get

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0 \quad \text{...(3.5)}$$

or $$\ddot{q} + 2b\dot{q} + \omega^2 q = 0 \qquad ...(3.6)$$

which is the differential equation that describes the damped oscillations. This is equivalent to Eqn. (3.2).

The general solution of Eqn. (3.6) can be written as (see Eqn. (3.10)) [$b < \omega$]

$$q = Q\, e^{-bt} \cos(\omega' t - \theta). \qquad ...(3.7)$$

Here, $$b = \frac{R}{2L},\ \omega = \frac{1}{\sqrt{Lc}},\ \omega' = \sqrt{\omega^2 - b^2}.$$

SOLVED PROBLEMS

1. *Obtain an expression for the displacement of the damped harmonic oscillator where the damping force is proportional to the velocity. Discuss the effect of the damping on the displacement and frequency of the oscillator.*

Solution

The differential equation of the damped harmonic motion is given by Eqn. (3.2):

$$\ddot{x} + 2\dot{x} + \omega^2 x = 0$$

where $(-2mb\dot{x}) = (-\beta\dot{x})$ is the damping force acting on the particle of mass m and ω is the natural frequency of the oscillator.

Let $x = \exp(\alpha t)$ be the trial solution of above equation. Then, we have

$$\alpha^2 + 2b\alpha + \omega^2 = 0.$$

The two roots of α are

$$\alpha_{1,2} = \frac{-2b \pm \sqrt{4b^2 - 4\omega^2}}{2} = -b \pm \sqrt{b^2 - \omega^2}$$

So, the general solution of Eqn. (3.2) is

$$x = e^{-bt}\left[A_1 \exp(\sqrt{b^2 - \omega^2}\, t) + A_2 \exp(-\sqrt{b^2 - \omega^2}\, t)\right] \qquad ...(3.8)$$

where A_1 and A_2 are two constants whose values can be determined from the initial conditions.

Case I: $b > \omega$ (*the damping force is large*)

The expression (3.8) for x represents a damped dead beat motion, the displacement x decreasing exponentially to zero (Fig. 3.1).

Case II: $b < \omega$ (*the damping is small*)

In this case $\sqrt{b^2 - \omega^2} = i\sqrt{\omega^2 - b^2} = i\omega'$ and Eqn. (3.8) becomes

$$x = e^{-bt}\,[B_1 \cos\omega' t + B_2 \sin\omega' t] \qquad ...(3.9)$$

where $\omega' = \sqrt{\omega^2 - b^2}$, $B_1 = A_1 + A_2$ and $B_2 = i\,(A_1 - A_2)$. If we write $B_1 = R\cos\theta$ and $B_2 = R\sin\theta$, Eqn. (3.9) reduces to

$$x = Re^{-bt}\cos(\omega' t - \theta) \qquad ...(3.10)$$

where $R = \sqrt{B_1^2 + B_2^2}$ and $\theta = \tan^{-1}(B_2/B_1)$

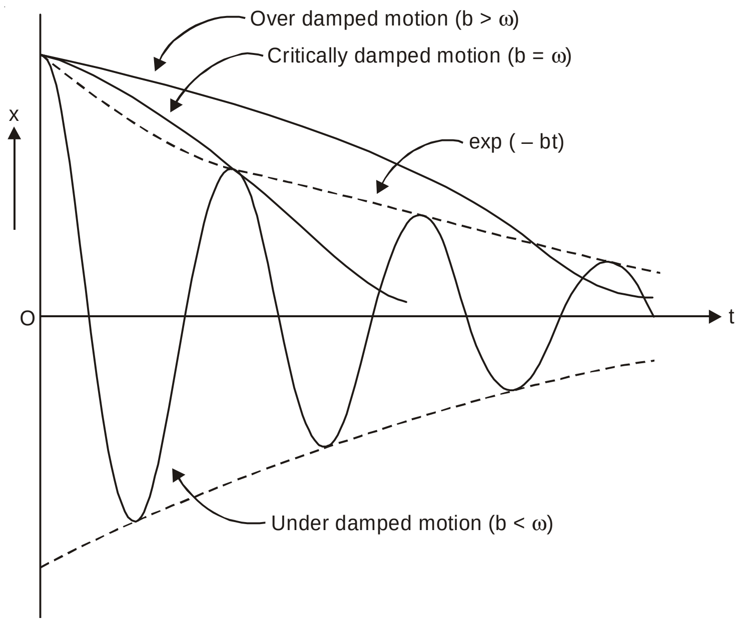

Fig. 3.1

Equation (3.10) gives a damped oscillatory motion (Fig. 3.1). Its amplitude R exp $(-bt)$ decreases exponentially with time. The time period of damped oscillation is

$$T = \frac{2\pi}{\omega'} = \frac{2\pi}{\sqrt{\omega^2 - b^2}} \qquad ...(3.11)$$

whereas the undamped time period is

$$T_0 = \frac{2\pi}{\omega}.$$

Thus the time period of damped oscillation is slightly greater than the undamped natural time period when $b \ll \omega$. In other words, the frequency of the damped oscillation $\omega' = \sqrt{\omega^2 - b^2}$ is less than the undamped natural frequency ω.

Let us consider the simple case in which $\theta = 0$ in Eqn. (3.10). Then $\cos \omega' t = \pm 1$ when $t = 0, t_1 = \frac{T}{2}, t_2 = T, t_3 = \frac{3T}{2}$ etc. Suppose that the values of x in both directions corresponding to these times are x_0, x_1, x_2, x_3 etc. so that

$$\begin{aligned} x_0 &= R \\ x_1 &= -Re^{-bT/2}, \\ x_2 &= Re^{-bT}, \\ x_3 &= -Re^{-3bT/2}, \\ &\vdots \end{aligned}$$

Considering the absolute values of the displacements, we get

$$\left|\frac{x_0}{x_1}\right| = \left|\frac{x_1}{x_2}\right| = \left|\frac{x_2}{x_3}\right| = \ldots = e^{bT/2}$$

The quantity $$\delta = 2 \ln \left|\frac{x_n}{x_{n+1}}\right| = bT = \frac{b}{\nu} \qquad ...(3.12)$$

is called logarithmic decrement. Here ν is the frequency of the damped oscillatory motion. The logarithmic decrement is the logarithm of the ratio of two successive maxima in one direction = ln (x_n/x_{n+2}). Thus the damping coefficient b can be found from an experimental measurement of consecutive amplitudes. Since

$$b = \frac{\beta}{2m},\ T = \frac{2\pi}{\sqrt{\omega^2 - b^2}} = \frac{1}{\nu} \text{ and } \omega^2 = k/m,$$

We have from Eqn. (3.12)

$$\delta = \frac{2\pi\beta}{\sqrt{4mk - \beta^2}}. \qquad ...(3.13)$$

The energy equation of the damped harmonic oscillator:

We can regard equation (3.10) as a cosine function whose amplitude R exp $(-\ bt)$ gradually decreases with time. For an undamped oscillator of amplitude R, the mechanical energy is constant and is given by $E = \frac{1}{2}kR^2$. If the oscillator is damped, the mechanical energy is not constant but decreases with time. For a damped oscillator the amplitude is R exp $(-\ bt)$ and the mechanical energy is

where $$E\ (0h) = \frac{1}{2}\ kR^2 \text{ is the initial mechanical energy.}$$

$$E\ (t) = \frac{1}{2}\ kR^2 \exp(-2bt)$$

Like the amplitude the mechanical energy decreases exponentially with time.

Case III: $b = \omega$ (*critically damped motion*)

When $b = \omega$, we get only one root $\alpha = -\ b$. One solution of Eqn. (3.2) is

$$x_1 = A_1 \exp(-bt)$$

and the other solution is

$$x_2 = A_2 t \exp(-bt).$$

So the general solution for critically damped motion (Fig. 3.1) is

$$x = e^{-bt}(A_1 + A_2 t) \qquad ...(3.14)$$

The motion is non-oscillatory and the particle approaches origin slowly.

2. *A particle of mass 3 moves along the x-axis attracted toward origin by a force whose magnitude is numerically equal to 12x. The particle is also subjected to a damping force whose magnitude is numerically equal to 12 times the instantaneous speed. If it is initially at rest at x = 10, find the position and the velocity of the particle at any time.*

Solution

The equation of motion of the particle is

$$3\ddot{x}\hat{i} = -12\dot{x}\hat{i} - 12x\hat{i}$$

or $$\ddot{x} + 4\ddot{x} + 4x = 0.$$

The solution of this equation is

$$x = e^{-2t}(A_1 + A_2 t).$$